

# CENTRALISERS OF SUBSYSTEMS OF FUSION SYSTEMS

J. SEMERARO

**ABSTRACT.** Let  $\mathcal{F}$  be a saturated fusion system on  $S$  and  $\mathcal{E}$  a normal subsystem of  $\mathcal{F}$  on  $T$ . We appeal to a result of Gross to give a construction of the  $\mathcal{F}$ -centraliser  $C_{\mathcal{F}}(\mathcal{E})$  which does not rely on Aschbacher's theory of normal maps. We also interpret the  $S$ -centraliser  $C_S(\mathcal{E})$  as the set of fixed points under a group action on a topological space.

Since their arrival at the end of the last century, saturated fusion systems have both provided a convenient framework for investigating  $p$ -local group theory and been objects of study in their own right. In the former case, it has been fruitful to translate notions from finite group theory into the context of fusion systems and look for generalisations of existing results. In this paper we take one such notion, the centraliser of a subgroup, and interpret it in the language of fusion systems.

Let  $\mathcal{F}$  be a fusion system on  $S$  and let  $\mathcal{E}$  be a subsystem of  $\mathcal{F}$  on  $T \leq S$ . A natural interpretation for ' $C_{\mathcal{F}}(\mathcal{E})$ ' is as the largest subsystem of  $\mathcal{F}$  containing all maps which 'centralise'  $\mathcal{E}$ . It is not yet clear what it should mean for a morphism to 'centralise' a subsystem, but there is a good notion of a centraliser of a subgroup so we may define  $C_S(\mathcal{E})$  (the group on which  $C_{\mathcal{F}}(\mathcal{E})$  acts) to be the set of all elements  $g$  with the property that  $\mathcal{E} \subseteq C_{\mathcal{F}}(\langle g \rangle)$ . One quickly realises that this set does not always form a group, even when  $\mathcal{E}$  is a weakly normal subsystem of  $\mathcal{F}$  (see Example 3.7). This suggests that a general  $\mathcal{F}$ -centraliser of a subsystem is too much to hope for. However, in [2], Aschbacher shows that when  $\mathcal{E}$  is *normal* in  $\mathcal{F}$ ,  $C_S(\mathcal{E})$  is an intersection of centralisers of constrained local subsystems, and does in fact form a group:

**Theorem A.** *Let  $T \leq S$  be a finite  $p$ -groups, let  $\mathcal{F}$  be a saturated fusion system on  $S$  and let  $\mathcal{E}$  be a normal subsystem of  $\mathcal{F}$  on  $T$ . Then  $C_S(\mathcal{E})$  is a group.*

Our proof of Theorem A is a modest improvement of that found in [3, Section 6]. We assume a key result in that paper ([3, Lemma 6.6.5]), but offer a simplified proof of the inductive step which shows that  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed, a necessary precursor to proving that  $C_S(\mathcal{E})$  is a group.

Assume now that  $\mathcal{E}$  is only weakly normal in  $\mathcal{F}$ . An alternative construction for  $C_S(\mathcal{E})$  comes from the theory of linking systems. In [6] Chermak uses his theory of partial groups to give a characterisation of  $C_S(\mathcal{E})$  as a subgroup of  $S$  which centralises objects in a linking system  $\mathcal{L}_0$  associated to  $\mathcal{E}$ . If  $\mathcal{L}$  is a linking system containing  $\mathcal{L}_0$  associated to  $\mathcal{F}$  then  $C_S(\mathcal{E})$  can also be viewed as the kernel of the action of  $\mathcal{L}$  on  $\mathcal{L}_0$  as the following result shows:

**Theorem B.** *Let  $T \leq S$  be finite  $p$ -groups,  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$  and  $(\mathcal{L}, \mathcal{L}_0)$  be a weakly normal pair of linking systems associated  $(\mathcal{F}, \mathcal{E})$ . The sequence*

$$1 \longrightarrow C_S(\mathcal{E}) \xrightarrow{\delta_S} \text{Aut}_{\mathcal{L}}(T) \xrightarrow{\gamma \mapsto c_\gamma} \text{Aut}_{\text{typ}}^I(\mathcal{L}_0) \longrightarrow \text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$$

1

is exact. In particular,  $C_S(\mathcal{E})$  is a group and  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed.

Here, the sets  $\text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  and  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$  consist of (classes) of isotypical self equivalences of  $\mathcal{L}_0$ , and such objects often have a topological interpretation. For example, when  $\mathcal{L} = \mathcal{L}_0$ ,  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L}) = \text{Out}_{\text{typ}}(\mathcal{L})$  is isomorphic to the group  $\text{Out}(|\mathcal{L}|_p^\wedge)$  of homotopy classes of maps from  $|\mathcal{L}|_p^\wedge$  to itself (see [5, Theorem 8.1]).

Lastly, we turn to the construction of the fusion system  $C_{\mathcal{F}}(\mathcal{E})$  in the case where  $\mathcal{E}$  is normal in  $\mathcal{F}$ . In [2]  $C_{\mathcal{F}}(\mathcal{E})$  is constructed using Aschbacher's theory of normal maps. Here, we propose an alternative proof of existence using the theory of fusion systems at index a power of  $p$ . Classical work of Gross [9] which uses the classification of finite simple groups reveals that whenever  $p$  is odd,  $H \trianglelefteq G$  are finite groups with  $O_{p'}(G) = 1$ ,  $S \in \text{Syl}_p(G)$  and  $T = S \cap H$ ,

$$(0.1) \quad O^p(C_G(T)) \cap S \leq C_G(H).$$

In the language of fusion systems, this implies that

$$\text{hfp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$$

where  $\mathcal{F}$  and  $\mathcal{E}$  are fusion systems associated to  $G$  and  $H$  respectively and  $\text{hfp}(-)$  is the *hyperfocal* subgroup. Fortunately (0.1) holds without the classification (and for  $p = 2$ ) when  $G$  and  $H$  are  $p$ -constrained, and we are able to use Aschbacher's local characterisation of  $C_S(\mathcal{E})$  to prove the following:

**Theorem C.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion system on  $(S, T)$ . Then*

$$\text{hfp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E}) \leq C_S(T).$$

*Thus there is a unique subsystem  $C_{\mathcal{F}}(\mathcal{E})$  on  $C_S(\mathcal{E})$  contained in  $C_{\mathcal{F}}(T)$  at index a power of  $p$ .*

The paper is structured as follows: In Section 1 we present a variety of basic results on fusion systems and linking systems which are needed in later sections. Section 2 deals with constrained fusion systems with a view to introducing the necessary local theory for normal subsystems. Theorems A and B are shown in Section 3 and Theorem C is shown in Section 4.

## 1. PRELIMINARIES

**1.1. Fusion systems.** We will assume the reader is familiar with the notion of a *fusion system*, and the terminology and results which can be found in [4, Part I]. The following proposition is fundamental to many of the arguments used in this paper, so we include it here.

**Proposition 1.1.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . The following are equivalent for each  $P \leq S$ .*

- (a)  *$P$  is fully  $\mathcal{F}$ -normalised.*
- (b)  *$P$  is fully  $\mathcal{F}$ -centralised and fully  $\mathcal{F}$ -automised.*
- (c) *For each  $Q \leq S$  and  $\varphi \in \text{Iso}_{\mathcal{F}}(Q, P)$  there exist  $\chi \in \text{Aut}_{\mathcal{F}}(P)$  and  $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$  such that  $\overline{\varphi}|_Q = \varphi \circ \chi$ .*

Next, we recall the definition of a normal subsystem of a fusion system. Given a fusion system  $\mathcal{F}$ ,  $\text{Aut}(\mathcal{F})$  is the subgroup of *fusion preserving maps* in  $\text{Aut}(S)$ , i.e. those which induce a functor from  $\mathcal{F}$  to itself. Also,  $T$  is *strongly  $\mathcal{F}$ -closed* if  $g\varphi \in T$  for each  $g \in T$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(\langle g \rangle, S)$ .

**Definition 1.2.** Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{E}$  be a subsystem of  $\mathcal{F}$  on  $T \leq S$ .

- (a)  $\mathcal{E}$  is  $\mathcal{F}$ -invariant if
  - (i)  $T$  is strongly  $\mathcal{F}$ -closed;
  - (ii)  $\text{Aut}_{\mathcal{F}}(T) \leq \text{Aut}(\mathcal{E})$ ; and
  - (iii) for each  $P \leq T$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(P, T)$ , there is  $\alpha \in \text{Aut}_{\mathcal{F}}(T)$  and  $\varphi_0 \in \text{Hom}_{\mathcal{E}}(P, T)$  such that  $\varphi = \varphi_0 \circ \alpha$ .
- (b)  $\mathcal{E}$  is *weakly normal* if  $\mathcal{E}$  is  $\mathcal{F}$ -invariant and saturated.
- (c)  $\mathcal{E}$  is (*strongly*) *normal* if  $\mathcal{E}$  is weakly normal and each  $\alpha \in \text{Aut}_{\mathcal{E}}(T)$  extends to a map  $\overline{\alpha} \in \text{Aut}_{\mathcal{F}}(TC_S(T))$  such that  $\overline{\alpha}|_T = \alpha$  and  $[\overline{\alpha}, C_S(T)] \leq Z(T)$ .

The definition of  $\mathcal{F}$ -invariance presented here depicts the ‘Frattini condition’ for morphisms. In fact, conditions (a)(ii) and (a)(iii) can be combined to produce the following equivalent, simpler definition for  $\mathcal{F}$ -invariance which we will occasionally use:

**Lemma 1.3.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . A subsystem  $\mathcal{E}$  of  $\mathcal{F}$  on  $T \leq S$  is  $\mathcal{F}$ -invariant if and only if*

- (i)  $T$  is strongly  $\mathcal{F}$ -closed; and
- (ii) for each  $P \leq T$ ,  $\varphi \in \text{Hom}_{\mathcal{E}}(P, T)$  and  $\psi \in \text{Hom}_{\mathcal{F}}(T, S)$ , we have

$$\psi^{-1}\varphi\psi \in \text{Hom}_{\mathcal{E}}(P\psi, T).$$

*Proof.* This is shown in [7, Theorem 5.43]. □

From now on, we will refer to  $(\mathcal{F}, \mathcal{E})$  as a (*weakly*) *normal pair* of fusion systems on  $(S, T)$  if  $\mathcal{E}$  is (weakly) normal in  $\mathcal{F}$ . Also, if  $G$  is a finite group,  $H$  is a normal subgroup of  $G$  and  $S$  is a Sylow  $p$ -subgroup of  $G$ , we refer to  $(G, H)$  as a *normal pair of finite groups with Sylow  $p$ -subgroup*  $(S, T)$  where  $T := S \cap H$ .

**Proposition 1.4.** *Let  $(G, H)$  be a normal pair of finite groups with Sylow  $p$ -subgroup  $(S, T)$ . Then  $(\mathcal{F}_S(G), \mathcal{F}_T(H))$  is a normal pair of fusion systems on  $(S, T)$ .*

*Proof.* See [7, Lemma 8.5]. □

We will need the notion of the *hyperfocal subgroup* of a fusion system, together with some basic facts concerning it.

**Definition 1.5.** Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $S$ . Then

$$\text{hfp}(\mathcal{F}) := \langle [R, \alpha] \mid R \leq S, \alpha \in O^p(\text{Aut}_{\mathcal{F}}(R)) \rangle.$$

**Lemma 1.6.** *Let  $G$  be a finite group with Sylow  $p$ -subgroup  $S$ . Then  $\text{hfp}(\mathcal{F}_S(G)) = O^p(G) \cap S$ .*

*Proof.* See [10, Section 1.1]. □

**Theorem 1.7.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$ . For each subgroup  $T$  satisfying  $\text{hfp}(\mathcal{F}) \leq T \leq S$  there exists a unique saturated subsystem of  $\mathcal{F}$  of  $p$ -power index on  $T$ .*

*Proof.* See [4, Theorem I.7.4]. □

We end this section with a result which concerns generation of normal subsystems. Note that a subgroup  $U$  of a group  $T$  is  *$T$ -centric* if  $C_T(U) = Z(U)$ .

**Theorem 1.8.** *Let  $T \leq S$  be finite  $p$ -groups and let  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Let  $\mathcal{U}$  be the set of all fully  $\mathcal{F}$ -normalised,  $T$ -centric subgroups of  $S$ . The following hold:*

- (a) *Each  $R \in \mathcal{U}$  is  $\mathcal{E}$ -centric.*
- (b)  $\mathcal{E} = \langle \text{Aut}_{\mathcal{E}}(U)^{\varphi|_U} \mid U \in \mathcal{U} \text{ and } \varphi \in \text{Aut}_{\mathcal{F}}(T) \rangle$ .

*Proof.* Let  $R$  be a fully  $\mathcal{F}$ -normalised and  $T$ -centric subgroup of  $S$  and let  $\varphi \in \text{Hom}_{\mathcal{F}}(R, S)$ . Since  $R$  is also fully  $\mathcal{F}$ -centralised by Proposition 1.1,  $\psi := \varphi^{-1}|_{R\varphi}$  extends to a map  $\bar{\psi} \in \text{Hom}_{\mathcal{F}}(R\varphi C_S(R\varphi), S)$  by the saturation of  $\mathcal{F}$ . Since  $T$  is strongly  $\mathcal{F}$ -closed,

$$|C_T(R\varphi)| = |C_T(R\varphi)\bar{\psi}| \leq |C_T(R)| = |Z(R)| = |Z(R\varphi)|,$$

and  $R$  is  $\mathcal{E}$ -centric. This proves (a). Part (b) is [2, Theorem 3(ii)].  $\square$

**1.2. Linking systems.** If  $S$  is a finite  $p$ -group and  $\mathcal{H}$  is a set of subgroups of  $S$ , let  $\mathcal{T}_{\mathcal{H}}(S)$  denote the category with object set  $\mathcal{H}$  and with  $\text{Hom}_{\mathcal{T}_{\mathcal{H}}(S)}(P, Q) := \{g \in S \mid g^{-1}Pg \leq Q\}$ . We begin with the definition of a linking system associated to a saturated fusion system.

**Definition 1.9.** ([1, Definition 1.9]) Let  $\mathcal{F}$  be a fusion system over a finite  $p$ -group  $S$ . A *linking system associated to  $\mathcal{F}$*  is a triple  $(\mathcal{L}, \delta, \pi)$  (referred to as ‘ $\mathcal{L}$ ’) where  $\mathcal{L}$  is a category with  $\text{Ob}(\mathcal{L}) \subseteq \text{Ob}(\mathcal{F})$  and where  $\delta$  and  $\pi$  are functors

$$\mathcal{T}_{\text{Ob}(\mathcal{L})}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}$$

which satisfy the following conditions:

- (a) (i)  $\text{Ob}(\mathcal{L})$  is closed under  $\mathcal{F}$ -conjugacy and taking overgroups in  $S$ .
- (ii) Each  $\mathcal{F}$ -centric,  $\mathcal{F}$ -radical subgroup is contained in  $\text{Ob}(\mathcal{L})$ .
- (iii) Each  $P \in \text{Ob}(\mathcal{L})$  is  $\mathcal{L}$ -isomorphic to a fully  $\mathcal{F}$ -centralised subgroup of  $S$ .
- (iv)  $\delta$  is the identity on objects and  $\pi$  is the inclusion on objects. For each  $P, Q \in \mathcal{H}$ , write  $\delta_{P,Q}(g)$  for the image of the morphism  $g \in \text{Hom}_{\mathcal{T}_{\text{Ob}(\mathcal{L})}(S)}(P, Q)$  under  $\delta$  and write  $\delta_P(g) := \delta_{P,P}(g)$  when  $Q = P$ . Similarly, write  $\pi_{P,Q}(\psi)$  for the image of  $\psi \in \text{Hom}_{\mathcal{L}}(P, Q)$  under  $\pi$ .
- (v) Whenever  $P, Q \in \text{Ob}(\mathcal{L})$  and  $P$  is fully  $\mathcal{F}$ -centralised,  $C_S(P)$  acts freely on  $\text{Hom}_{\mathcal{L}}(P, Q)$  via  $\delta_P$  by left composition and  $\pi_{P,Q}$  induces a bijection

$$\text{Hom}_{\mathcal{L}}(P, Q)/C_S(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

- (b) For each  $P, Q \in \text{Ob}(\mathcal{L})$  and  $g \in \text{Hom}_{\mathcal{T}_{\text{Ob}(\mathcal{L})}(S)}(P, Q)$ ,

$$\pi_{P,Q}(\delta_{P,Q}(g)) = c_g \in \text{Hom}_{\mathcal{F}}(P, Q).$$

- (c) For each  $\psi \in \text{Hom}_{\mathcal{L}}(P, Q)$  and  $g \in P$  the diagram

$$\begin{array}{ccc} P & \xrightarrow{\psi} & Q \\ \downarrow \delta_P(g) & & \downarrow \delta_Q(g\pi(\psi)) \\ P & \xrightarrow{\psi} & Q \end{array}$$

commutes.

Let  $\mathcal{L}$  be a linking system associated to a fusion system  $\mathcal{F}$  on a finite  $p$ -group  $S$ . A functor  $\alpha \in \text{Aut}(\mathcal{L})$  is *isotypical* if  $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$  for each  $P \in \text{Ob}(\mathcal{L})$ . Also, for each  $P, Q \in \text{Ob}(\mathcal{L})$ , write  $\iota_P^Q$  for the morphism  $\delta_{P,Q}(1) \in \text{Hom}_{\mathcal{L}}(P, Q)$  and call such a map an *inclusion from  $P$  to  $Q$* .

**Definition 1.10.** Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{L}$  be linking system associated  $\mathcal{F}$ .

- (a)  $\text{Aut}_{\text{typ}}^I(\mathcal{L})$  denotes the set of all isotypical self-equivalences of  $\mathcal{L}$  which send inclusions to inclusions.
- (b)  $\text{Out}_{\text{typ}}(\mathcal{L}) \subseteq \text{Out}(\mathcal{L})$  denotes the set of *outer automorphisms* consisting of representatives for the natural isomorphism classes of isotypical self-equivalences of  $\mathcal{L}$ .

Since we shall be interested in finding ‘inner automorphisms’ of  $\mathcal{L}$ , it will be important to understand how morphisms in  $\mathcal{L}$  restrict and extend. This information is provided by the following proposition, whose proof follows quickly from the axioms.

**Proposition 1.11.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{L}$  be a linking system associated to  $\mathcal{F}$ .*

- (a) *For each morphism  $\psi \in \text{Hom}_{\mathcal{L}}(P, Q)$ , and every  $P_0, Q_0 \in \text{Ob}(\mathcal{L})$  with  $P_0 \leq P$ ,  $Q_0 \leq Q$  and  $P_0\pi(\psi) \leq Q_0$ , there is a unique morphism  $\psi|_{P_0, Q_0} \in \text{Hom}_{\mathcal{L}}(P_0, Q_0)$  such that  $\iota_{P_0}^P \circ \psi = \psi|_{P_0, Q_0} \circ \iota_{Q_0}^Q$ .*
- (b) *Let  $P, Q, \overline{P}, \overline{Q} \in \text{Ob}(\mathcal{L})$  and  $\psi \in \text{Hom}_{\mathcal{L}}(P, Q)$  be such that  $P \trianglelefteq \overline{P}$  and  $Q \leq \overline{Q}$ . There is a morphism  $\overline{\psi} \in \text{Hom}_{\mathcal{L}}(\overline{P}, \overline{Q})$  such that  $\overline{\psi}|_{P, Q} = \psi$  if and only if*

$$\forall g \in \overline{P} \exists h \in \overline{Q} \text{ such that } \delta_P(g) \circ \psi \circ \iota_Q^{\overline{Q}} = \psi \circ \delta_{Q, \overline{Q}}(h).$$

*Furthermore if such a morphism exists, it is unique.*

*Proof.* Parts (a) and (b) are shown in [1, Proposition 1.11 (b) and (e)]. □

We can now introduce the concept of a ‘normal subsystem’ of a linking system.

**Definition 1.12.** ([1, Definition 1.9]) Let  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$ . We say that  $(\mathcal{L}, \mathcal{L}_0)$  is a *weakly normal pair of linking systems* associated to  $(\mathcal{F}, \mathcal{E})$  if the following hold:

- (a)  $\mathcal{L}$  is a linking system associated to  $\mathcal{F}$  and  $\mathcal{L}_0$  is a linking system associated to  $\mathcal{E}$  with  $\mathcal{L}_0 \subseteq \mathcal{L}$  and  $\text{Ob}(\mathcal{L}) = \{P \leq S \mid P \cap T \in \text{Ob}(\mathcal{L}_0)\}$ . Furthermore the structural functors

$$\mathcal{T}_{\text{Ob}(\mathcal{L}_0)}(T) \xrightarrow{\delta_0} \mathcal{L}_0 \xrightarrow{\pi_0} \mathcal{E}$$

are restrictions of the structural functors for  $\mathcal{L}$ .

- (b) (i) For all  $P, Q \in \text{Ob}(\mathcal{L}_0)$  and  $\psi \in \text{Hom}_{\mathcal{L}}(P, Q)$  there are morphisms  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$  and  $\psi_0 \in \text{Hom}_{\mathcal{L}_0}(P\pi(\gamma), Q)$  such that  $\psi = \gamma|_{P, P\pi(\gamma)} \circ \psi_0$ .
- (ii) For all  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$  and  $\psi \in \text{Mor}(\mathcal{L}_0)$ ,  $\gamma^{-1} \circ \psi \circ \gamma \in \text{Mor}(\mathcal{L}_0)$ .

We end with the construction of a ‘conjugation’ map  $\text{Aut}_{\mathcal{L}}(T) \longrightarrow \text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  for an arbitrary weakly normal pair  $(\mathcal{L}, \mathcal{L}_0)$  of linking systems associated to a weakly normal pair of fusion systems  $(\mathcal{F}, \mathcal{E})$  on  $(S, T)$ . For each  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$  let  $c_\gamma = c_\gamma|_{\mathcal{L}_0}$  be the functor from  $\mathcal{L}_0$  to itself defined by setting  $c_\gamma(P) := P\gamma := P\pi(\gamma)$  for  $P \in \text{Ob}(\mathcal{L}_0)$ .

$\text{Ob}(\mathcal{L}_0)$  and  $c_\gamma(\psi) := (\gamma|_{P, P\gamma})^{-1} \circ \psi \circ (\gamma|_{Q, Q\gamma}) \in \text{Hom}_{\mathcal{L}_0}(P\gamma, Q\gamma)$  for each  $\psi \in \text{Hom}_{\mathcal{L}_0}(P, Q)$ . (Here we use that  $\mathcal{L}_0$  is weakly normal in  $\mathcal{L}$ ).

**Lemma 1.13.** *Let  $(\mathcal{L}, \mathcal{L}_0)$  be a weakly normal pair of linking systems associated to a weakly normal pair of saturated fusion systems  $(\mathcal{F}, \mathcal{E})$  on  $(S, T)$ . Then  $c_\gamma|_{\mathcal{L}_0} \in \text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  for each  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$ .*

*Proof.* By Proposition 1.11 (a),  $\iota_P^Q \circ \gamma|_{Q, Q\gamma} = \gamma|_{P, P\gamma} \circ \iota_{P\gamma}^Q$  for each  $P, Q \in \text{Ob}(\mathcal{L}_0)$  so that  $c_\gamma$  clearly sends inclusions to inclusions. To see that  $c_\gamma$  is isotypical, note that for each  $P \in \text{Ob}(\mathcal{L}_0)$  and  $g \in P$ ,  $\delta_P(g) \circ \gamma|_{P, P\gamma} = \gamma|_{P, P\gamma} \circ \delta_{P\gamma}(g\pi(\gamma))$  by Definition 1.9 (c) for  $\mathcal{L}_0$ , so that  $c_\gamma(\delta_P(g)) = \delta_{P\gamma}(g\pi(\gamma))$ , as needed.  $\square$

## 2. CONSTRAINED FUSION SYSTEMS

Recall that a finite group  $G$  with  $O_{p'}(G) = 1$  is *p-constrained* if there exists a normal  $p$ -subgroup  $R$  with  $C_G(R) \leq R$ . This definition is extended to saturated fusion systems as follows.

**Definition 2.1.** A saturated fusion system  $\mathcal{F}$  on a finite  $p$ -group  $S$  is *constrained* if there exists an  $\mathcal{F}$ -centric subgroup of  $S$  which is normal in  $\mathcal{F}$ .

In this section, we develop the machinery required to prove that  $C_S(\mathcal{E})$  is a group (Theorem 3.6) whenever  $T \leq S$  are finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  is a strongly normal pair of fusion systems on  $(S, T)$ .

**2.1. The Subsystems  $\mathcal{E}(U)$  and  $\mathcal{D}(U)$ .** We begin with the definition from [2]. Let  $\mathcal{F}$  be a fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{E}$  be a saturated subsystem of  $\mathcal{F}$  on  $T \leq S$  such that  $T$  strongly  $\mathcal{F}$ -closed. For each fully  $\mathcal{F}$ -normalised subgroup  $U$  of  $T$ , write  $V := UC_T(U)$  and  $\overline{V} := VC_S(V)$ . Define

$$\mathcal{D}(U) := N_{N_{\mathcal{F}}(\overline{V})}(U) \text{ and } \mathcal{E}(U) := N_{N_{\mathcal{E}}(V)}(U).$$

The next lemma implies that  $\mathcal{E}(U)$  and  $\mathcal{D}(U)$  may be regarded as the largest canonically definable constrained fusion systems on  $N_T(U)$  and  $N_S(U)$  contained in  $\mathcal{E}$  and  $\mathcal{F}$  respectively.

**Lemma 2.2.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{E}$  be a saturated subsystem of  $\mathcal{F}$  on  $T \leq S$  such that  $T$  is strongly  $\mathcal{F}$ -closed. For each fully  $\mathcal{F}$ -normalised subgroup  $U$  of  $T$ , the following hold:*

- (a)  $\mathcal{D}(U)$  is a fusion system on  $N_S(U)$  and  $\mathcal{E}(U)$  is a fusion system on  $N_T(U)$ ;
- (b)  $\mathcal{D}(U)$  and  $\mathcal{E}(U)$  are both saturated, and  $V$  and  $\overline{V}$  are normal  $\mathcal{D}(U)$ - and  $\mathcal{E}(U)$ -centric subgroups of  $\mathcal{D}(U)$  and  $\mathcal{E}(U)$  respectively;
- (c)  $\mathcal{E}(U)$  is a subsystem of  $\mathcal{D}(U)$ ; and
- (d) if  $(\mathcal{F}, \mathcal{E})$  is a weakly normal pair then so is  $(\mathcal{D}(U), \mathcal{E}(U))$ .

*Proof.* We first prove (a). Since  $T$  is strongly  $\mathcal{F}$ -closed,  $V = UC_T(U)$  is a normal subgroup of  $N_S(U)$ . Now  $\overline{V} = VC_S(V) \trianglelefteq N_S(U)$  so that  $N_S(U) \leq N_S(\overline{V})$  and  $N_{N_S(\overline{V})}(U) = N_S(U) \cap N_S(\overline{V}) = N_S(U)$ . Clearly  $N_T(U) \leq N_T(V)$  so that  $N_{N_T(V)}(U) = N_T(V) \cap N_T(U) = N_T(U)$ , proving (a).

To see (b), first notice that we may write

$$\mathcal{D}(U) = N_{\mathcal{F}}^J(\overline{V}) \text{ and } \mathcal{E}(U) = N_{\mathcal{E}}^K(V)$$

where  $J := \{\alpha \in \text{Aut}_{\mathcal{F}}(\overline{V}) \mid \alpha|_U \in \text{Aut}(U)\}$  and  $K := \{\alpha \in \text{Aut}_{\mathcal{E}}(V) \mid \alpha|_U \in \text{Aut}(U)\}$ . Hence to prove that  $\mathcal{D}(U)$  and  $\mathcal{E}(U)$  are saturated, it suffices by [4,

Theorem I.5.5] to show that  $\overline{V}$  is fully  $J$ -normalised and  $V$  is fully  $K$ -normalised. Since  $T$  is strongly  $\mathcal{F}$ -closed,

$$N_S^{J^\phi}(\overline{V}\phi) = N_S(U\phi) \text{ and } N_T^{K^\psi}(V\psi) = N_T(U\psi) = N_S(U\psi) \cap T$$

for each  $\phi \in \text{Hom}_{\mathcal{F}}(\overline{V}, S)$  and  $\psi \in \text{Hom}_{\mathcal{F}}(V, S)$ . Hence both of these statements follow from the fact that  $U$  is fully  $\mathcal{F}$ -normalised. The second part of (b) is clear.

Next we prove (c). Clearly  $N_T(U) \leq N_S(U)$  so it suffices to show that each morphism in  $\mathcal{E}(U)$  is also a morphism in  $\mathcal{D}(U)$ . To this end, let  $P \leq N_T(U)$  and  $\varphi \in \text{Hom}_{\mathcal{E}(U)}(P, N_T(U))$ . We need to show that  $\varphi$  extends to a map  $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(P\overline{V}, S)$  with  $\overline{\varphi}|_U \in \text{Aut}(U)$ , since any map with this property clearly also fixes  $\overline{V}$ . By definition,  $\varphi$  extends to a map  $\tilde{\varphi} \in \text{Hom}_{\mathcal{E}}(PV, N_T(U))$  with  $\tilde{\varphi}|_U \in \text{Aut}(U)$ . Hence we may replace  $P$  by  $PV$  and  $\varphi$  by  $\tilde{\varphi}$  if necessary and assume that  $V \leq P$ .

Suppose the identity

$$\text{Aut}_{C_S(V)}(P) \leq O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P))$$

holds. By Proposition 1.1 (c) there is some  $\beta \in \text{Hom}_{N_{\mathcal{F}}(U)}(N_{N_S(U)}(P\varphi), N_S(U))$  such that  $P' := P\varphi\beta$  is fully  $N_{\mathcal{F}}(U)$ -normalised.  $P'$  is also fully  $N_{\mathcal{F}}(U)$ -automised so that  $O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P')) \leq \text{Aut}_{N_S(U)}(P')$ . Now

$$\text{Aut}_{C_S(V)}(P)^{\varphi\beta} \leq O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P))^{\varphi\beta} = O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P')) \leq \text{Aut}_{N_S(U)}(P').$$

This means that  $C_S(V) \leq N_{\varphi\beta}$  so there is some  $\psi \in \text{Hom}_{N_{\mathcal{F}}(U)}(PC_S(V), N_S(U))$  such that  $\psi|_P = \varphi \circ \beta|_P$ . Set  $\overline{\varphi} := \psi \circ \beta^{-1} \in \text{Hom}_{N_{\mathcal{F}}(U)}(PC_S(V), N_S(U))$ . Since  $\overline{\varphi}|_P = \varphi$ ,  $\overline{\varphi}$  is a map with the required properties.

It remains to prove that  $\text{Aut}_{C_S(V)}(P) \leq O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P))$ . Since  $P \leq N_T(U)$  and  $T$  is strongly  $\mathcal{F}$ -closed,  $[P, C_S(V)] \leq C_S(V) \cap T = C_T(V) = Z(V) \leq P$ , since  $V$  is  $T$ -centric. Hence  $C_S(V) \leq N_{N_S(U)}(P)$  and  $\text{Aut}_{C_S(V)}(P)$  must centralise the factors in the  $\text{Aut}_{N_{\mathcal{F}}(U)}(P)$  invariant chain

$$1 \leq Z(V) \leq P.$$

Thus [8, Corollary 5.3.3] implies that  $\text{Aut}_{C_S(V)}(P) \leq O_p(\text{Aut}_{N_{\mathcal{F}}(U)}(P))$ , as needed.

Finally, we prove that (d) holds. To do this, we use the characterisation of weak normality provided by Lemma 1.3. Let  $\varphi \in \text{Hom}_{\mathcal{E}(U)}(P, N_T(U))$  and  $\alpha \in \text{Hom}_{\mathcal{D}(U)}(N_T(U), N_S(U))$ . There exist elements

$$\overline{\varphi} \in \text{Hom}_{\mathcal{E}}(PV, T) \text{ and } \overline{\alpha} \in \text{Hom}_{\mathcal{F}}(N_T(U)VC_S(V), S),$$

(where  $V = UC_T(U)$ ) extending  $\varphi$  and  $\alpha$  respectively and such that  $U\overline{\varphi} = U = U\overline{\alpha}$ . Hence  $\overline{\alpha}^{-1}|_{P\alpha V} \circ \overline{\varphi} \circ \overline{\alpha}|_{PV}$  lies in  $\text{Hom}_{\mathcal{E}}(P\alpha V, N_T(U))$  (since  $(\mathcal{F}, \mathcal{E})$  is a weakly normal pair) and extends  $\alpha^{-1} \circ \varphi \circ \alpha \in \text{Hom}_{\mathcal{E}(U)}(P\alpha, N_T(U))$ , as needed.  $\square$

One of the major advantages in working with normal pairs of fusion systems  $(\mathcal{F}, \mathcal{E})$  is that they behave very well in the case where  $\mathcal{F}$  and  $\mathcal{E}$  are both constrained, as the next result shows. Recall that by [4, Theorem I.4.9], a constrained fusion system  $\mathcal{F}$  is realisable as the fusion system of a  $p$ -constrained group  $G$ .

**Theorem 2.3.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Assume further that  $\mathcal{F}$  is constrained. Let  $G$  be  $p$ -constrained group with  $O_{p'}(G) = 1$  which realises  $\mathcal{F}$ . Then there exists a  $p$ -constrained normal subgroup  $H$  of  $G$  which realises  $\mathcal{E}$ .*

*Proof.* See [2, Theorem 1].  $\square$

Since the converse to Theorem 2.3 holds by Proposition 1.4, the lattice of normal subsystems of  $\mathcal{F}$  is in one to one correspondence with the lattice of normal subgroups of  $G$ . This has a large number of applications. For example, Lemma 2.2 (d) can be extended to an analogous statement for normal pairs  $(\mathcal{F}, \mathcal{E})$ .

**Theorem 2.4.** *Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $S$  and let  $\mathcal{E}$  be a saturated subsystem of  $\mathcal{F}$  on  $T \leq S$  such that  $T$  is strongly  $\mathcal{F}$ -closed. Let  $U$  be a fully  $\mathcal{F}$ -normalised subgroup of  $T$ . If  $(\mathcal{F}, \mathcal{E})$  is a normal pair then so is  $(\mathcal{D}(U), \mathcal{E}(U))$ .*

*Proof.* By Lemma 2.2,  $(\mathcal{D}(U), \mathcal{E}(U))$  is a weakly normal pair so it remains to check that condition (c) in Definition 1.2 holds. Let  $U$  be a counterexample with  $n := |T : U|$  chosen as small as possible. If  $n = 1$  then  $U = T$ ,  $(\mathcal{D}(T), \mathcal{E}(T)) = (N_{\mathcal{F}}(TC_S(T)), N_{\mathcal{E}}(T))$  and since  $\text{Aut}_{\mathcal{E}(T)}(T) = \text{Aut}_{\mathcal{E}}(T)$  and  $\text{Aut}_{\mathcal{D}(T)}(TC_S(T)) = \text{Aut}_{\mathcal{F}}(TC_S(T))$  the fact that  $(\mathcal{D}(T), \mathcal{E}(T))$  is a normal pair follows from the fact that  $(\mathcal{F}, \mathcal{E})$  is. Hence we may assume that  $n > 1$ . Write  $R := N_T(U)$  and fix an element  $\alpha \in \text{Aut}_{\mathcal{E}(U)}(R)$ . Let  $\varphi \in \text{Hom}_{\mathcal{F}}(R, T)$  be chosen so that  $R\varphi$  is fully  $\mathcal{F}$ -normalised and  $\varphi$  extends to a map  $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_S(R), N_S(R\varphi))$  by Proposition 1.1 (c). Write  $\beta := \varphi^{-1} \circ \alpha \circ \varphi \in \text{Aut}_{\mathcal{F}}(R\varphi)$ , so  $\beta$  extends to an element  $\bar{\beta} \in \text{Aut}_{\mathcal{F}}(R\varphi C_S(R\varphi))$  (since  $\mathcal{F}$  is saturated). Since  $T$  is strongly  $\mathcal{F}$ -closed,  $(R\varphi C_S(R\varphi) \cap T)\beta = R\varphi C_T(R\varphi)$  which implies  $\bar{\beta} \in \text{Mor}(\mathcal{D}(R\varphi))$ .

By the minimality of  $n$ ,  $(\mathcal{D}(R\varphi), \mathcal{E}(R\varphi))$  is a normal pair of fusion systems so that by Theorem 2.4 there exists a normal pair  $(G(R\varphi), H(R\varphi))$  of finite groups which realises  $(\mathcal{D}(R\varphi), \mathcal{E}(R\varphi))$ . Since  $U \leq T$  is fully  $\mathcal{F}$ -normalised, it is also fully  $\mathcal{E}$ -normalised so that  $UC_T(U)$  and  $R = N_T(U)$  are  $\mathcal{E}$ -centric by Lemma ?? . Hence  $R\varphi$  is a normal  $\mathcal{E}(R\varphi)$ -centric subgroup, so that  $C_H(R\varphi) = Z(R\varphi)$ . Since  $\beta \in \text{Mor}(\mathcal{E}(R\varphi))$  there is  $h \in H(R\varphi)$  such that  $\beta = c_h|_{R\varphi}$ . We have

$$[h, C_S(R\varphi)] \leq C_H(R\varphi) = Z(R\varphi),$$

so  $\bar{\beta} = c_h|_{R\varphi C_S(R\varphi)}$  satisfies  $[\bar{\beta}, C_S(R\varphi)] \leq Z(R\varphi)$ . Define

$$\bar{\alpha} := \bar{\varphi}|_{RC_S(R)} \circ \bar{\beta} \circ \varphi^{-1}|_{R\varphi C_S(R\varphi)}$$

and note that  $\bar{\alpha}$  extends  $\alpha$  and satisfies  $[\bar{\alpha}, C_S(R)] \leq Z(R)$ . In particular  $\mathcal{E}(U)$  is a normal subsystem of  $\mathcal{D}(U)$ , a contradiction.  $\square$

Combining Theorems 2.3 and 2.4 yields the following result.

**Corollary 2.5.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Let  $U$  be a  $\mathcal{F}$ -normalised subgroup of  $T$ . There exist finite groups  $G(U)$  and  $H(U)$  with  $H(U) \trianglelefteq G(U)$  such that*

$$\mathcal{D}(U) = \mathcal{F}_{N_S(U)}(G(U)) \text{ and } \mathcal{E}(U) = \mathcal{F}_{N_T(U)}(H(U)).$$

*Proof.* See [2, Theorem 3].  $\square$

**2.2. The Subsystem  $\mathcal{D}(U, X)$ .** In the proof of Theorem 3.6, it will be important to keep track of subgroups  $X$  which centralise  $\mathcal{E}$  and fully  $\mathcal{F}$ -normalised  $T$ -centric subgroups  $U$  of  $S$  simultaneously. It is unfortunate that for this purpose, the subsystems  $\mathcal{E}(U)$  and  $\mathcal{D}(U)$  will not suffice. To overcome this problem, we adapt the definition of  $\mathcal{D}(U)$  as follows.



**Definition 2.6.** Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$ . Fix  $X \leq C_S(T)$  and write  $\mathcal{U}$  for the set of all fully  $\mathcal{F}$ -normalised,  $T$ -centric subgroup  $U$  of  $T$ . Define:

$$U_X := UXC_S(UX) \text{ and } \mathcal{D}(U, X) := N_{\mathcal{F}}^L(U_X),$$

where  $L := \{\varphi \in \text{Aut}_{\mathcal{F}}(U_X) \mid \varphi|_{UX} \in \text{Aut}_{\mathcal{F}}(UX)\}$ .

Observe that  $\mathcal{D}(U, 1) = \mathcal{D}(U)$  when  $U$  is  $T$ -centric. The next result shows that  $\mathcal{E}(U)$  is a subsystem of  $\mathcal{D}(U, X)$  and provides us with an analogue of Lemma 2.2 for the pair  $(\mathcal{D}(U, X), \mathcal{E}(U))$ .

**Theorem 2.7.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$ . For each  $X \leq C_S(T)$  and each  $U \in \mathcal{U}$  for which  $UX$  is fully  $N_{\mathcal{F}}(U)$ -normalised, the following hold:*

- (a)  $\mathcal{D}(U, X)$  is a saturated subsystem of  $N_{\mathcal{F}}(U)$  on  $N_S(UX)$  and  $U_X$  is a normal  $\mathcal{D}(U, X)$ -centric subgroup of  $\mathcal{D}(U, X)$ ;
- (b)  $\mathcal{E}(U)$  is a subsystem of  $\mathcal{D}(U, X)$ ; and
- (c) If  $(\mathcal{F}, \mathcal{E})$  is a normal pair then so is  $(\mathcal{D}(U, X), \mathcal{E}(U))$ .

*Proof.* Observe that by the Dedekind identity,

$$U \leq T \cap UX \leq T \cap U_X \leq T \cap UC_S(U) = U(T \cap C_S(U)) = U.$$

We first prove (a). By definition, each element of  $\text{Mor}(\mathcal{D}(U, X))$  extends to a map  $\psi$  which leaves  $UX$  invariant. Since  $T$  is strongly closed,

$$U\psi = (UX)\psi \cap T = UX \cap T = U,$$

so that  $\mathcal{D}(U, X) \subseteq N_{\mathcal{F}}(U)$ . Also,  $U_X \trianglelefteq N_S(UX)$  so we must have

$$N_{N_S(U)}^L(U_X) = N_S(UX) \cap N_S(UX) = N_S(UX).$$

To see that  $\mathcal{D}(U, X)$  is saturated, it suffices (by [4, Theorem I.5.5]) to show that  $U_X$  is fully  $L$ -normalised in  $N_{\mathcal{F}}(U)$ . Let  $\varphi \in \text{Hom}_{N_{\mathcal{F}}(U)}(U_X, N_S(U))$ . Then  $N_{N_S(U)}^L(U_X) = N_S(UX)$  and  $N_{N_S(U)}^{L^\varphi}(U_X\varphi) = N_{N_S(U)}((UX)\varphi)$  so saturation follows from the fact that  $UX$  is fully  $N_{\mathcal{F}}(U)$ -normalised. It is clear that  $U_X$  is a normal  $\mathcal{D}(U, X)$ -centric subgroup of  $\mathcal{D}(U, X)$  and this completes the proof of (a).

Next we prove (b). Let  $P \leq N_T(U)$  and  $\varphi \in \text{Hom}_{\mathcal{E}(U)}(P, N_T(U))$ . We need to show that  $\varphi$  extends to a map  $\overline{\varphi}$  such that  $[UX, \overline{\varphi}] \leq UX$ . By Corollary 2.5,

$$(\mathcal{D}(U), \mathcal{E}(U)) = (\mathcal{F}_{N_S(U)}(G(U)), \mathcal{F}_{N_T(U)}(H(U)))$$

for some normal pair  $(G(U), H(U))$  of finite groups. Hence there is  $g \in H(U)$  such that  $\varphi = c_g \in \text{Hom}_{G(U)}(P, N_T(U))$  and

$$[UX, \varphi] = [UX, c_g] \leq UC_S(U) \cap H(U) = UC_S(U) \cap T = U \leq UX,$$

as required.

An identical argument to that which proves Lemma 2.2 (d) shows that the pair  $(\mathcal{D}(U, X), \mathcal{E}(U))$  is weakly normal. To make this explicit, consider a pair of morphisms  $\varphi \in \text{Hom}_{\mathcal{E}(U)}(P, N_T(U))$  and  $\alpha \in \text{Hom}_{\mathcal{D}(U, X)}(N_T(U), N_S(UX))$ . There exist elements

$$\overline{\varphi} \in \text{Hom}_{\mathcal{E}}(PU, T) \text{ and } \overline{\alpha} \in \text{Hom}_{\mathcal{F}}(N_T(U)U_X, S)$$

extending  $\varphi$  and  $\alpha$  respectively such that  $U\overline{\varphi} = U = U\overline{\alpha}$ . Hence  $\overline{\alpha}^{-1}|_{P\alpha U \circ \overline{\varphi} \circ \overline{\alpha}}|_{PU}$  lies in  $\text{Hom}_{\mathcal{E}}(P\alpha U, N_T(U))$  and extends  $\alpha^{-1} \circ \varphi \circ \alpha \in \text{Hom}_{\mathcal{E}(U)}(P\alpha, N_T(U))$ , as needed. Finally, observe that

$[N_{H(U)}(N_T(U)), C_S(N_T(U))] \leq H(U) \cap C_S(N_T(U)) = C_{H(U)}(N_T(U)) = Z(N_T(U))$ , and since there is  $g \in N_{H(U)}(N_T(U))$  such that  $\varphi = c_g$  for each  $\varphi \in \text{Aut}_{\mathcal{E}(U)}(N_T(U))$ , we must have that  $(\mathcal{D}(U, X), \mathcal{E}(U))$  is a normal pair. This completes the proof of (c).  $\square$

Theorem 2.7 naturally gives rise to the following analogue of Corollary 2.5 for the pair  $(\mathcal{D}(U, X), \mathcal{E}(U))$ .

**Corollary 2.8.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$ . For each  $X \leq C_S(T)$  and each  $U \in \mathcal{U}$  for which  $UX$  is fully  $N_{\mathcal{F}}(U)$ -normalised there exist finite groups  $G(U, X)$  and  $H(U, X)$  with  $H(U, X) \trianglelefteq G(U, X)$  such that*

$$\mathcal{D}(U, X) = \mathcal{F}_{N_S(UX)}(G(U, X)) \text{ and } \mathcal{E}(U) = \mathcal{F}_{N_T(U)}(H(U, X)).$$

Furthermore,  $H(U)$  is a subgroup of  $G(U, X)$  and

$$C_{G(U, X)}(H(U)) \cap N_S(UX) = C_{G(U, X)}(H(U, X)) \cap N_S(UX).$$

*Proof.* The first statement is an immediate consequence of Theorem 2.3. The second is [3, Lemma 6.3.5].  $\square$

### 3. THE $S$ -CENTRALISER $C_S(\mathcal{E})$

**3.1. Centralisers of Local Subsystems.** Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion systems on  $(S, T)$ . We are now almost in a position to prove that the set

$$C_S(\mathcal{E}) := \{g \in C_S(T) \mid \mathcal{E} \subseteq C_{\mathcal{F}}(\langle g \rangle)\}$$

is (1) a group and (2) strongly  $\mathcal{F}$ -closed. We prove (2) in Theorem 3.4 first and use this to deduce (1) in Theorem 3.6. The first step is to gain some control over the way in which groups which centralise  $\mathcal{E}$  behave under  $\mathcal{F}$ -conjugation. The next lemma shows that the obstruction to  $C_S(\mathcal{E})$  being strongly  $\mathcal{F}$ -closed is that not all maps defined on subgroups of  $C_S(\mathcal{E})$  extend to maps on  $T$ .

**Lemma 3.1.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$ . Let  $X \leq C_S(T)$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(X, S)$ . The following hold:*

- (a) *If  $\varphi$  extends to a map  $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(XT, S)$  then  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  if and only if  $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$ .*
- (b) *If  $X\varphi$  is fully  $\mathcal{F}$ -centralised then  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  if and only if  $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$ . In particular, if  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  then there is some fully  $\mathcal{F}$ -normalised  $\mathcal{F}$ -conjugate,  $Y$  of  $X$  such that  $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$ .*
- (c) *If  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  then  $N_T(X\varphi) = C_T(X\varphi)$ . In particular,  $T \cap (X\varphi C_T(X\varphi))\gamma = C_T(X\varphi)\gamma$  for each  $\gamma \in \text{Hom}_{\mathcal{F}}(X\varphi C_T(X\varphi), S)$ .*

*Proof.* First we prove (a). Let  $\alpha \in \text{Hom}_{\mathcal{E}}(P, T)$ . If  $\alpha$  is a morphism in  $C_{\mathcal{F}}(X)$  then  $\alpha$  extends to a map  $\overline{\alpha} \in \text{Hom}_{\mathcal{F}}(PX, TX)$  with  $\overline{\alpha}|_X = \text{Id}_X$ . Hence  $\overline{\varphi}^{-1} \circ \overline{\alpha} \circ \overline{\varphi} \in \text{Hom}_{\mathcal{F}}((PX)\overline{\varphi}, S)$  extends  $\beta := \overline{\varphi}|_{P\varphi}^{-1} \circ \alpha \circ \overline{\varphi}|_{P\alpha}$  and restricts to the identity on  $X\varphi$  so that  $\beta$  is a morphism in  $C_{\mathcal{F}}(X\varphi)$ . Now  $\varphi|_T \in \text{Aut}_{\mathcal{F}}(T) \leq \text{Aut}(\mathcal{E})$  (since  $\mathcal{E}$  is

weakly normal in  $\mathcal{F}$ ), so  $\beta$  is also a morphism in  $\mathcal{E}$  and  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  if and only if  $\mathcal{E} \subseteq C_{\mathcal{F}}(X\varphi)$ , as needed.

Next, we prove (b). Note that since  $\mathcal{F}$  is saturated, there is  $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$  extending  $\varphi$ . Since  $T \leq C_S(X) \leq N_{\varphi}$ , the first statement follows from (a). If  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$  and  $\alpha \in \text{Hom}_{\mathcal{F}}(X, S)$  is chosen so that  $X\alpha$  is fully  $\mathcal{F}$ -normalised, then by Proposition 1.1  $X\alpha$  is also fully  $\mathcal{F}$ -centralised. Thus the second statement follows from the first.

Finally, let  $Y$  be a fully  $\mathcal{F}$ -normalised  $\mathcal{F}$ -conjugate of  $X\varphi$  satisfying  $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$ . By Proposition 1.1, there exists a map  $\beta \in \text{Hom}_{\mathcal{F}}(N_S(X\varphi), N_S(Y))$  such that  $X\varphi\beta = Y$ . Now  $N_T(X\varphi)\beta \leq N_T(Y) = C_T(Y)$  and on composing both sides of this equation with  $\beta^{-1}$ , the first statement in (c) follows. The second statement follows immediately from the first since  $T$  is strongly  $\mathcal{F}$ -closed.  $\square$

Thus, a consequence of Lemma 3.1 (a) is that  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed whenever  $T$  is a normal subgroup of  $\mathcal{F}$ . To handle the general case, we will need to proceed inductively over the index of  $C_T(X)$  for each subgroup  $X$  which centralises  $\mathcal{E}$ . To do this, we will show that all such subgroups  $X$  centralise all  $T$ -centric, fully  $\mathcal{F}$ -normalised subgroups  $U$ . The first step is to find a condition which ensures that  $X$  centralises  $H(U)$  for any given pair  $(X, U)$ .

**Definition 3.2.** Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion systems on  $(S, T)$ . For each  $X, U \leq S$ , we say that  $(X, U)$  *satisfies  $\dagger$*  if

- (i)  $X \leq C_S(T)$  and  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ ;
- (ii)  $U$  is fully  $\mathcal{F}$ -normalised and  $T$ -centric; and
- (iii) for all  $Y \in X^{\mathcal{F}}$  with  $|C_T(Y)| > |U|$ ,  $\mathcal{E} \subseteq C_{\mathcal{F}}(Y)$ .

The next result implies that  $X$  centralises  $H(U)$  if  $(X, U)$  satisfies  $\dagger$ . Since the argument given in [3] is fairly long and technical we will not reproduce it here.

**Lemma 3.3.** Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion systems on  $(S, T)$ . If  $(X, U)$  satisfies  $\dagger$  then  $X \leq C_{N_S(U)}(H(U))$ .

*Proof.* This is shown in [3, Theorem 6.5].  $\square$

We are finally in a position to show that  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed. The argument we present is an improvement of that found in [3, Theorem 6.6].

**Theorem 3.4.** Let  $T \leq S$  be finite  $p$ -groups and let  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Then  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed.

*Proof.* In what follows, write  $\mathcal{C}$  for the set of all subgroups  $X$  of  $S$  which satisfy  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ . It will be enough to show that  $\mathcal{C}$  is closed under  $\mathcal{F}$ -conjugacy. To this end, assume that there is some  $Y \in \mathcal{C}$  such that  $Y^{\mathcal{F}} \not\subseteq \mathcal{C}$ . We may choose  $Y$  to be fully  $\mathcal{F}$ -normalised by Lemma 3.1 (b). We claim that there exists a morphism  $\gamma \in \text{Hom}_{\mathcal{F}}(Y, S)$  and a subgroup  $R$  of  $T$  such that  $(Y\gamma, R)$  satisfies  $\dagger$ .

Amongst all elements of  $Y^{\mathcal{F}}$  which do not lie in  $\mathcal{C}$ , let  $X$  be chosen so that  $n := |T : C_T(X)|$  is as small as possible. Since  $Y$  is fully  $\mathcal{F}$ -normalised, by Proposition 1.1 there is some  $\alpha \in \text{Hom}_{\mathcal{F}}(XC_T(X), S)$  with  $X\alpha = Y$ . Hence, by Lemma 3.1 (a) we must have  $n > 1$ . Set  $Q := C_T(X)\alpha$  and  $W := YQ$ .

By Proposition 1.1, we may choose  $\gamma \in \text{Hom}_{\mathcal{F}}(N_S(W), N_S(W\gamma))$  such that  $W\gamma$  is fully  $\mathcal{F}$ -normalised. Since  $Y \in \mathcal{C}$ ,  $T$  centralises  $Y$  and since  $N_T(Q) \leq N_S(W)$  we must have  $N_T(Q)\gamma \leq C_T(Y\gamma)$ . Lastly,  $Q < T$  implies that

$$|C_T(X)| = |Q| < |N_T(Q)| \leq |C_T(Y\gamma)|,$$

so that by the minimality of  $n$ ,  $Y\gamma \in \mathcal{C}$ .

Since  $W\gamma$  is fully  $\mathcal{F}$ -normalised, we may choose  $\beta \in \text{Hom}_{\mathcal{F}}(N_S(XP), N_S(W\gamma))$  so that  $(XP)\beta = W\gamma$ . Define

$$\varphi := \beta^{-1} \circ \alpha \circ \gamma \in \text{Aut}_{\mathcal{F}}(W\gamma),$$

noting that  $X\beta\varphi = Y\gamma$ . We show that  $X\beta$  does not centralise  $N_T(W\gamma)$ . First observe that  $X$  does not centralise  $N_T(XC_T(X))$ . To see this, since  $C_T(X) < T$  we have  $XC_T(X) < N_{XT}(XC_T(X))$  and there exist  $x \in X$  and  $t \in T$  such that  $xt \in N_{XT}(XC_T(X)) \setminus XC_T(X)$ . Clearly  $x \in XC_T(X)$  so we must have  $t \in T \setminus XC_T(X)$  and in particular  $C_T(X) \leqslant XC_T(X) < N_T(XC_T(X))$ . Now suppose that  $X\beta \leqslant C_S(N_T(W\gamma))$ . This implies

$$X \leqslant C_S(N_T(W\gamma))\beta^{-1} \leqslant C_S(N_T(W\gamma)\beta^{-1})$$

and since  $N_T(XP)\beta \leqslant N_T((XP)\beta) = N_T(W\gamma)$  we must have  $C_S(N_T(W\gamma)\beta^{-1}) \leqslant C_S(N_T(XP))$  which delivers the required contradiction.

Set  $R := Q\gamma C_T(Q\gamma)$ . Proposition 1.1 implies that there is  $\rho \in \text{Hom}_{\mathcal{F}}(N_S(R), S)$  such that  $R\rho$  is fully  $\mathcal{F}$ -normalised. By Lemma 3.1 (c),  $N_S(W\gamma) \leqslant N_S(Q\gamma) \leqslant N_S(R)$  so that  $N_S(W\gamma)\rho \leqslant N_S(W\gamma\rho)$  and  $W\gamma\rho$  is fully  $\mathcal{F}$ -normalised since  $W\gamma$  is. Thus on replacing  $\gamma$  by  $\gamma \circ \rho$  if necessary, we may assume that  $R$  is fully  $\mathcal{F}$ -normalised. Hence by [4, Theorem I.5.5],  $N_{\mathcal{F}}(R)$  is saturated which means that there is some  $\rho' \in \text{Hom}_{N_{\mathcal{F}}(R)}(N_{N_S(R)}(RY\gamma), N_S(R))$  such that  $(RY\gamma)\rho'$  is fully  $N_{\mathcal{F}}(R)$ -normalised by Proposition 1.1. Since  $RY\gamma = Q\gamma C_T(Q\gamma)Y\gamma = W\gamma C_T(Q\gamma)$ , we have  $N_S(W\gamma) \leqslant N_{N_S(R)}(RY\gamma)$  so  $N_S(W\gamma)\rho' \leqslant N_S(W\gamma\rho')$  and  $W\gamma\rho'$  is fully  $\mathcal{F}$ -normalised. Thus on replacing  $\gamma$  by  $\gamma \circ \rho'$  if necessary, we may assume that  $RY\gamma$  is fully  $N_{\mathcal{F}}(R)$ -normalised. In particular, we have shown that (by the minimality of  $n$ ) the pair  $(Y\gamma, R)$  satisfies  $\dagger$ .

Since  $W\gamma$  is fully  $\mathcal{F}$ -normalised, there is  $\psi \in \text{Aut}_{\mathcal{F}}(W\gamma C_S(W\gamma))$  which extends  $\varphi$ . Also,  $R\psi = R$  by Lemma 3.1 (c) (note that  $R \leqslant C_S(W\gamma)$  and  $W\gamma\psi = W\gamma$ ). Therefore  $\psi$  leaves  $W\gamma R = Y\gamma R$  invariant and then also  $Y\gamma R C_S(Y\gamma R)$  invariant. This shows that  $\psi \in \text{Mor}(\mathcal{D}(R, Y\gamma))$ . Since  $RY\gamma$  is fully  $N_{\mathcal{F}}(R)$ -normalised,  $\mathcal{D}(R, Y\gamma)$  is saturated and there exist finite groups  $G = G(R, Y\gamma)$  and  $H = H(R, Y\gamma)$  which respectively realise  $\mathcal{D}(R, Y\gamma)$  and  $\mathcal{E}(R, Y\gamma)$  by Corollary 2.8. By Lemma 3.3,  $Y\gamma \leqslant C_G(H(R)) = C_G(H)$  and we may also choose  $g \in G$  such that  $c_g|_V = \psi|_V$ . Finally,

$$Y\gamma = X\beta\varphi = X\beta\psi = (X\beta)^g \leqslant C_{N_S(RY\gamma)}(H)$$

and since  $H \trianglelefteq G$ , we have  $X\beta \leqslant C_{N_S(RY\gamma)}(H) \leqslant C_S(N_T(W\gamma))$  which implies that  $N_T(W\gamma) \leqslant C_T(X\beta)$ , a contradiction.  $\square$

We now combine Lemma 3.3 with Theorems 1.8 and 3.4 to prove that  $C_S(\mathcal{E})$  is a group. It will be useful to make the following definition, in order to render the result more explicit.

**Definition 3.5.** Let  $T \leqslant S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Let  $\mathcal{U}$  be the set of all fully  $\mathcal{F}$ -normalised,  $T$ -centric subgroups of  $S$ . Define

$$X(\mathcal{E}) := \bigcap_{\varphi \in \text{Aut}_{\mathcal{F}}(TC_S(T))} \left( \bigcap_{U \in \mathcal{U}} C_{N_S(U)}(H(U)) \right) \varphi.$$

**Theorem 3.6.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of saturated fusion systems on  $(S, T)$ . Then  $C_S(\mathcal{E}) = X(\mathcal{E})$  and in particular,  $C_S(\mathcal{E})$  is a group.*

*Proof.* First we prove that  $C_S(\mathcal{E}) \leq X$ , where  $X := X(\mathcal{E})$ . Let  $R \leq C_S(T)$  be such that  $\mathcal{E} \subseteq C_{\mathcal{F}}(R)$ . By Theorem 3.4,  $(R\varphi, U)$  satisfies  $\dagger$  for all fully  $\mathcal{F}$ -normalised  $T$ -centric subgroups  $U$  of  $T$  and each  $\varphi \in \text{Aut}_{\mathcal{F}}(TC_S(T))$ . Hence by Lemma 3.3,  $R\varphi \leq C_{G(U)}(H(U))$  and  $R \leq X$ . Conversely note that  $\text{Aut}_{H(U)}(U)^\varphi = \text{Aut}_{H(U)}(U\varphi) \leq \text{Aut}_{C_{G(U)}(X)}(U\varphi)$  for each fully  $\mathcal{F}$ -normalised  $T$ -centric subgroup  $U$  of  $T$  and  $\varphi \in \text{Aut}_{\mathcal{F}}(TC_S(T))$ . Since  $\mathcal{E}$  is generated by the groups  $\text{Aut}_{H(U)}(U)^\varphi$  by Theorem 1.8, we must have  $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ , as needed.  $\square$

**3.2.  $C_S(\mathcal{E})$  as the Kernel of an Action.** We motivate Theorem B with an example which proves that there exist weakly normal pairs  $(\mathcal{F}, \mathcal{E})$  of fusion systems on  $(S, T)$  for which  $C_S(\mathcal{E})$  is not a group. Thus, a generalisation of Theorem 3.6 to the case where  $(\mathcal{F}, \mathcal{E})$  is weakly normal is too much to hope for.

**Example 3.7.** Let  $H := H_1 \times H_2 \times H_3$  with  $H_i \cong A_4$  for  $1 \leq i \leq 3$  be a direct product of three copies of the alternating group on 4 letters. For  $1 \leq i \leq 3$  pick  $S_i \in \text{Syl}_2(H_i)$  so that  $S_1 \times S_2 \times S_3 \in \text{Syl}_2(H)$  and let  $X_i = \langle x_i \rangle$  be a group of order 3 which acts on  $S_i$  in such a way that  $H_i = S_i \rtimes X_i$  for each  $i$ . Set  $X := \langle x_1x_2, x_1x_3 \rangle \cong C_3 \times C_3$ ,  $G := SX$  and let

$$\mathcal{F} := \mathcal{F}_S(G), \quad \mathcal{F}_1 := \mathcal{F}_{S_1S_2}(\langle S_1, S_2, x_1x_2 \rangle) \text{ and } \mathcal{F}_2 := \mathcal{F}_{S_1S_3}(\langle S_1, S_3, x_1x_3 \rangle).$$

Clearly  $\mathcal{F}_i$  is normal in  $\mathcal{F}$  for  $i = 1, 2$  and hence the fusion system  $\mathcal{E} := \mathcal{F}_1 \cap \mathcal{F}_2$  on  $S_1$ , whose morphisms consist of those in both  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , is an  $\mathcal{F}$ -invariant subsystem on  $S_1$  (intersections of  $\mathcal{F}$ -invariant subsystems are  $\mathcal{F}$ -invariant). Furthermore  $\mathcal{E} = \mathcal{F}_{S_1}(H_1)$  so  $\mathcal{E}$  is saturated by Theorem 1.4 and hence weakly normal. However,  $\mathcal{E}$  is not normal in  $\mathcal{F}$ . To see this, let  $\varphi \in \text{Aut}_{\mathcal{E}}(S_1)$  be the map induced by conjugation by  $x_1$  so that  $\varphi$  extends to maps  $\varphi_1 = c_{x_1x_2}$  and  $\varphi_2 = c_{x_1x_3} \in \text{Aut}_{\mathcal{F}}(S)$ . Observe that  $\varphi_i$  does not act trivially on  $C_S(S_i)/Z(S_i) = S/S_1 = S_2 \times S_3$  for  $i = 1, 2$ , proving the claim.

Now, for  $i = 2, 3$ ,  $\mathcal{E} \subseteq C_{\mathcal{F}}(S_i) = \mathcal{F}_{4-i} \times \mathcal{F}_{S_i}(S_i)$ . Note that  $\mathcal{E}$  is not contained in  $C_{\mathcal{F}}(S_2 \times S_3) = \mathcal{F}_S(S)$  so there is no unique maximal subgroup of  $S$  which centralises  $\mathcal{E}$ . In particular,  $C_S(\mathcal{E})$  is not a group.

Since objects  $Q$  in a linking system  $\mathcal{L}$  associated to  $\mathcal{F}$  must satisfy  $C_{\mathcal{F}}(Q) = \mathcal{F}_{C_S(Q)}(C_S(Q))$  ([4, Proposition III.4.7]) we see that  $S_1 \notin \text{Ob}(\mathcal{L})$  and there can exist no weakly normal pair  $(\mathcal{L}, \mathcal{L}_0)$  of linking systems associated to  $(\mathcal{F}, \mathcal{E})$ . We will show that this is always the case for a weakly normal pair  $(\mathcal{F}, \mathcal{E})$  of fusion systems when  $C_S(\mathcal{E})$  is not a group, which follows from the existence of a certain exact sequence into which  $C_S(\mathcal{E})$  fits. To construct this sequence, we will need a generalisation of the set  $\text{Out}_{\text{typ}}(\mathcal{L})$ .

**Definition 3.8.** Let  $\mathcal{C} \subseteq \mathcal{D}$  be finite categories. Two elements  $F$  and  $G$  of  $\text{Aut}(\mathcal{C})$  are  $\mathcal{D}$ -naturally isomorphic if there exists a map  $\eta$  from  $F$  to  $G$  which associates to each  $A \in \text{Ob}(\mathcal{C})$  a morphism  $\eta_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$  such that for all  $\varphi \in \text{Hom}_{\mathcal{C}}(A, B)$ , the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \downarrow F(\varphi) & & \downarrow G(\varphi) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

commutes.

**Definition 3.9.** Let  $(\mathcal{L}, \mathcal{L}_0)$  be a pair of linking systems (associated to some pair of fusion systems). The set  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$  is the set of equivalence classes of  $\text{Aut}_{\text{typ}}(\mathcal{L}_0)$  under  $\mathcal{L}$ -natural isomorphism.

Our motivation for proving the next result came from [1, Lemma 1.14], which follows by specialising to the case  $\mathcal{E} = \mathcal{F}$ .

**Theorem 3.10.** *Let  $T \leq S$  be finite  $p$ -groups,  $(\mathcal{F}, \mathcal{E})$  be a weakly normal pair of fusion systems on  $(S, T)$  and  $(\mathcal{L}, \mathcal{L}_0)$  be a weakly normal pair of linking systems associated  $(\mathcal{F}, \mathcal{E})$ . The sequence*

$$1 \longrightarrow C_S(\mathcal{E}) \xrightarrow{\delta_S} \text{Aut}_{\mathcal{L}}(T) \xrightarrow{\gamma \mapsto c_\gamma} \text{Aut}_{\text{typ}}^I(\mathcal{L}_0) \longrightarrow \text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$$

*is exact. In particular,  $C_S(\mathcal{E})$  is a group and  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed.*

*Proof.* By [1, Lemma 1.14],  $\text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  is a group and so  $C_S(\mathcal{E})$  and  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$  are both groups provided the sequence is exact. First we show that the natural homomorphism from  $\text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  to  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$  is onto. This amounts to showing that each  $\alpha \in \text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  is  $\mathcal{L}$ -naturally isomorphic to an isotypical equivalence which sends inclusions to inclusions. However, by [1, Lemma 1.14] any such  $\alpha$  is  $\mathcal{L}_0$ -naturally isomorphic to an isotypical equivalence, establishing the claim.

Now suppose that  $\alpha \in \text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  is in the kernel of the map from  $\text{Aut}_{\text{typ}}^I(\mathcal{L}_0)$  to  $\text{Out}_{\text{typ}}(\mathcal{L}_0, \mathcal{L})$ . Then  $\alpha$  is  $\mathcal{L}$ -naturally isomorphic to the identity functor via some set of maps  $\eta$  consisting of isomorphisms  $\eta_P \in \text{Iso}_{\mathcal{L}}(P, \alpha(P))$  and for each  $\psi \in \text{Hom}_{\mathcal{L}_0}(P, Q)$ ,  $\eta_P \circ \alpha(\psi) = \psi \circ \eta_Q$ . Since  $\alpha$  is isotypical  $\alpha(\iota_P^T) = \iota_{\alpha(P)}^T$  so that  $\eta_P = \eta_T|_{P, \alpha(P)}$  and this shows that  $\alpha$  is conjugation by  $\eta_T \in \text{Aut}_{\mathcal{L}}(T)$ . Conversely, if  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$  then  $c_\gamma$  is  $\mathcal{L}$ -naturally isomorphic to  $\text{Id}_{\mathcal{L}_0}$  via the set of maps  $\{\gamma|_{P, P\pi(\gamma)}\}_{P \in \text{Ob}(\mathcal{L}_0)}$ .

Next we show that for each  $\gamma \in \text{Aut}_{\mathcal{L}}(T)$ ,  $c_\gamma = \text{Id}_{\mathcal{L}_0}$  if and only if  $\gamma \in \delta_S(C_S(\mathcal{E}))$ . If  $c_\gamma = \text{Id}_{\mathcal{L}_0}$  then since  $\gamma^{-1} \circ \delta_T(g) \circ \gamma = \delta_T(g)$  for all  $g \in S$ , we must have  $\pi(\gamma) = \text{Id}_T$  by Definition 1.9 (c) for  $\mathcal{L}$  and the fact that  $\delta_T$  is injective. Hence by Definition 1.9 (a)(v) for  $\mathcal{L}$ , there is some  $a \in C_S(T)$  such that  $\gamma = \delta_T(a)$ . Now, for each  $P, Q \in \text{Ob}(\mathcal{L}_0)$  and  $\psi \in \text{Mor}_{\mathcal{L}_0}(P, Q)$ ,  $\psi \circ \delta_Q(a) = \delta_P(a) \circ \psi$  so that by Proposition 1.11 (b), there is some  $\bar{\psi} \in \text{Mor}_{\mathcal{L}}(\langle P, a \rangle, \langle Q, a \rangle)$  such that  $\bar{\psi}|_{P, Q} = \psi$  (note that  $a \in C_S(T) \leq C_S(P)$ ). By Definition 1.9 (c) for  $\mathcal{L}$  again and the injectivity of  $\delta$ ,  $\pi(\bar{\psi})(a) = a$ . Hence each morphism in  $\mathcal{E}$  extends to one which fixes  $a$  and  $a \in C_S(\mathcal{E})$ . Conversely if  $\gamma = \delta_S(a)$  for some  $a \in C_S(\mathcal{E})$  then each  $\psi \in \text{Mor}(\mathcal{L}_0)$  extends to some  $\bar{\psi}$  with  $\pi(\bar{\psi})(a) = a$ . By Definition 1.9 (c) for  $\mathcal{L}$ ,  $\bar{\psi}$  commutes with  $\delta_S(a)$  and we have  $c_\gamma(\psi) = \psi$ . Hence  $c_\gamma = \text{Id}_{\mathcal{L}_0}$ , as required.

It remains to prove the last statement. First we show that  $C_S(\mathcal{E})$  is strongly  $\mathcal{F}$ -closed. Let  $a \in C_S(\mathcal{E})$  and  $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$  with  $a \in P$ . Let  $\psi \in \text{Hom}_{\mathcal{L}}(P, S)$  be such that  $\pi(\psi) = \varphi$  and note that by Definition 1.9 (c) for  $\mathcal{L}$ , we have  $\psi^{-1} \circ \delta_P(a) \circ \psi = \delta_S(a\varphi)$ . This implies that  $c_{\delta_S(a\varphi)} = c_\psi^{-1} \circ \text{Id}_{\mathcal{L}_0} \circ c_\psi = \text{Id}_{\mathcal{L}_0}$  (since  $c_{\delta_P(a)} = \text{Id}_{\mathcal{L}_0}$ ) and  $a\varphi \in C_S(\mathcal{E})$  as needed.  $\square$

#### 4. THE $\mathcal{F}$ -CENTRALISER $C_{\mathcal{F}}(\mathcal{E})$

Let  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion systems on  $(S, T)$ . A definition of  $C_{\mathcal{F}}(\mathcal{E})$ , due to Aschbacher, has already appeared in Chapter 6 of [3], and the results in this section rely on some of the machinery developed there. Our approach does not

however use the theory of normal maps (see [2, Section 7]). Instead we rely on a group theoretic result, which is a corollary to the following theorem of Gross in [9].

**Theorem 4.1.** *Let  $G$  be a  $p$ -constrained finite group with Sylow  $p$ -subgroup  $S$ . Assume that  $O_{p'}(G) = 1$  and write*

$$C := C_{\text{Aut}(G)}(S) = \{\varphi \in \text{Aut}(G) \mid \varphi|_S = \text{Id}_S\}.$$

*Then  $C$  has a normal  $p$ -complement.*

*Proof.* This follows from (1) and (2) in [9, Section 5]: if  $G$  is a  $p$ -constrained, minimal counterexample to the lemma, (1) and (2) imply that  $O_p(G) = 1$ , a contradiction. Note that the assumption  $p > 2$  is not used anywhere in these two steps.  $\square$

**Corollary 4.2.** *Let  $T \leq S$  be finite  $p$ -groups and  $(G, H)$  be a normal pair of  $p$ -constrained finite groups with Sylow  $p$ -subgroup  $(S, T)$ . If  $O_{p'}(G) = 1$  then*

$$O^p(C_G(T)) \cap S \leq C_G(H).$$

*Proof.* Let  $C_{\text{Aut}_G(H)}(T)$  be the image of  $C_G(T)$  under the natural map

$$\Phi : C_G(T) \longrightarrow \text{Aut}_G(H).$$

Theorem 4.1 implies that  $C_{\text{Aut}_G(H)}(T)$  has a normal  $p$ -complement. In particular,  $O^p(C_{\text{Aut}_G(H)}(T))$  is a  $p'$ -group. Since

$$(O^p(C_G(T)) \cap S)C_G(H)/C_G(H) \leq O^p(C_G(T))C_G(H)/C_G(H) \leq O^p(C_G(T)/C_G(H)),$$

and this latter group is a  $p'$ -group, we must have  $O^p(C_G(T)) \cap S \leq C_G(H)$ , as needed.  $\square$

As an immediate consequence of Lemma 1.6 and Theorem 2.3 we obtain:

$$\text{hnp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$$

whenever  $(\mathcal{F}, \mathcal{E})$  is a normal pair of constrained fusion systems. In particular, there is a saturated fusion system  $C_{\mathcal{F}}(\mathcal{E})$  on  $C_S(\mathcal{E})$  contained in  $C_{\mathcal{F}}(T)$  by Theorem 1.7. Our goal is to prove this statement for *all* normal pairs of fusion systems by combining Corollary 4.2 with Theorem A. The main difficulty lies in showing that  $C_{\mathcal{F}}(T)$ -automorphisms of  $p'$  order extend (in  $\mathcal{F}$ ) to  $\mathcal{D}(U)$ -morphisms for each fully  $\mathcal{F}$ -normalised  $T$ -centric subgroup  $U$ . We overcome this problem by first considering normal chains of subgroups from  $U$  to  $T$ :

**Definition 4.3.** Let  $\mathcal{F}$  be a fusion system on  $\mathcal{F}$ , let  $T \leq S$  be a strongly  $\mathcal{F}$ -closed subgroup and  $X \leq C_S(T)$ . A chain of subgroups

$$\mathcal{C}(U) := U = U_0 < U_1 < \cdots < U_n = T$$

is *strongly  $(\mathcal{F}, X)$ -normalised* if  $U_{i+1} = N_T(U_i)$ ,  $U_i$  is fully  $\mathcal{F}$ -normalised and  $U_i X$  is fully  $N_{\mathcal{F}}(U_i)$ -normalised for each  $0 \leq i < n$ .

For the remainder of this section we will assume that  $T \leq S$  are finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  is a normal pair of fusion systems on  $(S, T)$ . Write  $\mathcal{U}$  for the set of all fully  $\mathcal{F}$ -normalised  $T$ -centric subgroups of  $T$ . We would like to assert that each  $W \in \mathcal{U}$  is  $\mathcal{F}$ -conjugate to a subgroup  $U$  which affords a strongly  $(\mathcal{F}, X)$ -normalised chain  $\mathcal{C}(U)$ . For this, we state without proof the following lemma of Aschbacher which essentially says that local subsystems behave well under taking normalisers:

**Lemma 4.4.** *Let  $X \leq C_S(T)$  and  $U \in \mathcal{U}$  and write  $Q := N_T(U)$ . There exists  $\alpha \in \text{Hom}_{\mathcal{F}}(N_S(Q), S)$  with the following properties:*

- (a)  $U\alpha, Q\alpha \in \mathcal{U}$  and  $(XU)\alpha, (XQ)\alpha$  are fully  $N_{\mathcal{F}}(U\alpha)$ -normalised.
- (b)  $\alpha$  extends to a map  $\check{\alpha} : G(U, X) \rightarrow G(U\alpha, X\alpha)$  which satisfies

$$N_{G(Q\alpha, X\alpha)}(U\alpha) = N_{G(U, X)}(Q)\check{\alpha}$$

*Proof.* This follows from parts (6), (7) and (8) of [3, Lemma 6.6.3].  $\square$

**Corollary 4.5.** *Let  $\mathcal{F}$  be a saturated fusion system on  $\mathcal{F}$ , let  $T \leq S$  be a strongly  $\mathcal{F}$ -closed subgroup and  $X \leq C_S(T)$ . For each  $W \in \mathcal{U}$  there exists  $U \in W^{\mathcal{F}}$  such that  $\mathcal{C}(U)$  is strongly  $(\mathcal{F}, X)$ -normalised.*

*Proof.* Assuming the lemma is false, let  $W$  be a counterexample with  $n := |T : W|$  as small as possible and write  $Q := N_T(W)$ . Since  $T$  is fully  $\mathcal{F}$ -normalised,  $n > 1$ .

By Lemma 4.4 (a), there is some  $\alpha \in \text{Hom}_{\mathcal{F}}(N_S(Q), S)$  such that  $W\alpha, Q\alpha \in \mathcal{U}$  and  $(XW)\alpha, (XQ)\alpha$  are fully  $N_{\mathcal{F}}(W\alpha)$ -normalised. Now set  $U := W\alpha$  and the result is proven.  $\square$

Now, for each  $U \in \mathcal{U}$  and  $X \leq C_S(T)$ , define:

$$C(U, X) := C_{G(U, X)}(N_T(U)) \text{ and } K(U, X) := O^p(C(U, X)).$$

We can apply Lemma 4.4 to show that  $K(U, X)$  is isomorphic to  $K(T, X)$  for each  $U \in \mathcal{U}$  in a very strong sense.

**Lemma 4.6.** [2, Lemma 6.10] *Let  $X \leq C_S(T)$  and  $U \in \mathcal{U}$  be such that*

$$\mathcal{C}(U) := U = U_0 < U_1 < \dots < U_n = T$$

*is a strongly  $(\mathcal{F}, X)$ -normalised chain. For each  $0 \leq j < n$ , there exists an isomorphism*

$$\theta_j : N_{G(U_j, X)}(U_{j+1}) \rightarrow N_{G(U_{j+1}, X)}(U_j)$$

*which acts like the identity on  $N_S(U_j)$ . Furthermore,  $K(U_j, X)\theta_j \dots \theta_n = K(T, X)$ .*

*Proof.* The existence of isomorphisms  $\theta_j$  with the prescribed property is an immediate consequence of Lemma 4.4 above. It remains to prove the last statement. Writing  $K_j := K(U_j, X)$  and  $G_j := G(U_j, X)$  we have  $K_j\theta_j \leq C_{G_{j+1}}(U_{j+1})$  so that  $[K_j\theta_j, U_{j+2}] \leq C_{G_{j+1}}(U_{j+1})$  (since  $U_{j+2}$  normalises  $U_{j+1}$ ). Conversely  $K_{j+1}\theta_j^{-1}$  centralises  $U_{j+1}$  so we must have  $K_j\theta_j = K_{j+1}$ , as required.  $\square$

Finally, we obtain the required result from which we can immediately deduce Theorem C.

**Lemma 4.7.** *Let  $X \leq C_S(T)$  and  $U \in \mathcal{U}$  be such that*

$$\mathcal{C}(U) := U = U_0 < U_1 < \dots < U_n = T$$

*is a strongly  $(\mathcal{F}, X)$ -normalised chain. We have*

$$\text{hnp}(C_{\mathcal{F}}(T)) \leq \text{hnp}(C_{\mathcal{D}(U, X)}(N_T(U))).$$

*Proof.* It suffices to show that for each  $X \leq C_S(T)$  and  $\alpha \in O^p(\text{Aut}_{C_{\mathcal{F}}(T)}(X))$ ,  $\alpha \in O^p(\text{Aut}_{\mathcal{D}(U, X)}(X))$ . Suppose we have shown that

$$(4.1) \quad O^p(\text{Aut}_{C_{\mathcal{F}}(T)}(X)) = \text{Aut}_{K(T, X)}(X).$$



Then since  $X \leq N_S(U_j)$  for each  $j$ , Lemma 4.6 implies that  $\text{Aut}_{K(T,X)}(X) = \text{Aut}_{K(U,X)}(X)$  and the result follows by the definition of  $K(U, X)$ . It thus remains to prove (4.1). Clearly  $\text{Aut}_{K(T,X)}(X) \leq O^p(\text{Aut}_{C_{\mathcal{F}}(T)}(X))$  by definition. Conversely, each  $p'$ -automorphism  $\alpha \in \text{Aut}_{C_{\mathcal{F}}(T)}(X)$  extends to a morphism  $\bar{\alpha} \in \text{Aut}_{\mathcal{F}}(XT)$  which fixes  $T$ . Thus  $\bar{\alpha} = c_g$  for some  $g \in K(T, X)$  and

$$O^p(\text{Aut}_{C_{\mathcal{F}}(T)}(X)) \leq \text{Aut}_{K(T,X)}(X),$$

as required.  $\square$

**Proposition 4.8.** *For each  $U \in \mathcal{U}$  and  $X \leq C_S(T)$  we have  $\text{hnp}(C_{\mathcal{F}}(T)) \leq C_{G(U,X)}(H(U, X)) \cap N_S(UX)$ . In particular,  $\text{hnp}(C_{\mathcal{F}}(T)) \leq C_S(\mathcal{E})$ .*

*Proof.* By Corollary 4.5 there is some  $\beta \in \text{Hom}_{N_{\mathcal{F}}(U)}(N_S(UX), S)$  so that  $\mathcal{C}(U\beta)$  is strongly  $(\mathcal{F}, X)$ -normalised. By [2, Lemma 7.2.4],  $\beta|_{UX} = c_b$  for some  $b \in G(U)$ . By [3, Lemma 6.3.7] and since  $C_{G(U,X)}(H(U, X)) \cap N_S(UX) = C_{G(U,X)}(H(U)) \cap N_S(U)$  (Corollary 2.8), the result holds for  $(U, X)$  if it holds for  $(U\beta, X\beta)$ . Thus we may replace  $U$  by  $U\beta$  and assume that  $\mathcal{C}(U)$  is fully  $(\mathcal{F}, X)$ -normalised. By Lemma 4.7 we have

$$\text{hnp}(C_{\mathcal{F}}(T)) \leq \text{hnp}(C_{\mathcal{D}(U,X)}(N_T(U))) = O^p(C_{G(U,X)}(N_T(U)) \cap N_S(UX)),$$

where the last equality follows from Lemma 1.6. Finally by Corollary 4.2 we have  $O^p(C_{G(U,X)}(N_T(U)) \cap N_S(UX)) \leq C_{G(U,X)}(H(U, X))$ , which proves the proposition.  $\square$

**Theorem 4.9.** *Let  $T \leq S$  be finite  $p$ -groups and  $(\mathcal{F}, \mathcal{E})$  be a normal pair of fusion system on  $(S, T)$ . There is a unique subsystem  $C_{\mathcal{F}}(\mathcal{E})$  on  $C_S(\mathcal{E})$  contained in  $C_{\mathcal{F}}(T)$  at index a power of  $p$ .*

*Proof.* This follows immediately from Proposition 4.8 and Theorem 1.7.  $\square$

## REFERENCES

- [1] Kasper Andersen, Bob Oliver, and Joana Ventura. Reduced, tame and exotic fusion systems. *Proc. London Math. Soc.*, 105 (1):87–152, 2012.
- [2] Michael Aschbacher. Normal subsystems of fusion systems. *Proc. London Math. Soc.*, 97:239–271, 2008.
- [3] Michael Aschbacher. The generalized fitting subsystem of a fusion system. *Mem. Amer. Math. Soc.*, 209, 2011.
- [4] Michael Aschbacher, Radha Kessar, and Bob Oliver. *Fusion Systems in Algebra and Topology*. Cambridge University Press, 2011.
- [5] Carles Broto, Ran Levi, and Bob Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 26:779–856, 2003.
- [6] Andrew Chermak. The normal structure of linking systems. preprint, 2012.
- [7] David A. Craven. *The Theory of Fusion Systems*. Cambridge studies in advanced mathematics; 131, 2011.
- [8] Daniel Gorenstein. *Finite groups*. Harper and Row, 1968.
- [9] Fletcher Gross. Automorphisms which centralize a sylow  $p$ -subgroup. *J. Alg.*, 77:202–233, 1982.
- [10] Lluís Puig. The hyperfocal subalgebra of a block. *Invent. math.*, 141:365–397, 2000.

HEILBRONN INSTITUTE FOR MATHEMATICAL RESEARCH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRISTOL, U.K.

*E-mail address:* js13525@bristol.ac.uk